Hamilton cycles in hypergraphs

Richard Mycroft

A famous theorem of Dirac states that any graph $\mathcal{G}$ on $n \geq 3$ vertices with minimum degree $\delta(\mathcal{G}) \geq n/2$ contains a Hamilton cycle, i.e. a cycle containing every vertex. We will consider analogues of this theorem for hypergraphs.

One way of defining cycles in graphs is to say that a graph $\mathcal{C}$ is a cycle if there is a cyclic ordering of the vertices of $\mathcal{C}$ such that every edge consists of 2 consecutive vertices in this ordering, and every pair of adjacent edges intersect in one vertex. We define cycles in hypergraphs by extending this idea: a k-graph $\mathcal{C}'$ is an ℓ-cycle (where $1 \leq \ell \leq k-1$) if there is a cyclic ordering of the vertices of $\mathcal{C}'$ such that every edge consists of k consecutive vertices and every pair of adjacent edges intersect in precisely ℓ vertices. Just as in the graph case, a Hamilton ℓ-cycle in a k-graph $\mathcal{H}$ is a subgraph of $\mathcal{H}$ which contains every vertex of $\mathcal{H}$ and which is an ℓ-cycle.

Similarly, in a graph $\mathcal{G}$ the degree of a vertex is the number of edges of $\mathcal{G}$ containing that vertex. We extend this idea to a k-graph $\mathcal{H}$ by defining the degree $d(S)$ of a set S of $k-1$ vertices to be the number of edges containing S. As for graphs, the minimum degree $\delta(\mathcal{H})$ is the minimum of $d(S)$ taken over all sets S of $k-1$ vertices.

So, for any $1 \leq \ell \leq k-1$, to generalise Dirac's theorem we wish to answer the following question: what minimum value of $\delta(\mathcal{H})$ ensures that a k-graph $\mathcal{H}$ contains a Hamilton ℓ-cycle? Building on work by Rödl, Ruciński and Szemerédi, and also Hàn and Schacht, we shall show how to obtain an asymptotic solution for all such k and ℓ.