Combinatorial theorems in sparse random sets

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The famous theorem of Szemerédi says that for any $k \in \mathbb{N}$ and $\delta > 0$ there exists n_0 such that if $n \geq n_0$ then any subset A of the set $[n] = \{1, 2, \cdots, n\}$ of size $|A| \geq \delta n$ contains an arithmetic progression of length k. The question we look at is when does such a theorem hold in a random set. More precisely, we say that a set X is (δ, k)-Szemerédi if every subset Y of X that contains at least $\delta|X|$ elements contains an arithmetic progression of length k. We prove that there is a threshold at about $p = n^{-1/(k-1)}$ where the probability that the random set $[n]_p$ is (δ, k)-Szemerédi changes from being almost surely 0 to almost surely 1.

There are many other similar problems within combinatorics. For example, Turán's theorem and Ramsey's theorem may be relativised, but until now the precise probability thresholds were not known. Our method seems to apply to all such questions, in each case giving the correct threshold. This is joint work with Tim Gowers.